

FRAMES ADAPTED TO A PHASE-SPACE COVER

MONIKA DÖRFLER AND JOSÉ LUIS ROMERO

ABSTRACT. We construct frames adapted to a given cover of the time-frequency or time-scale plane. The main feature is that we allow for quite general and possibly irregular covers. The frame members are obtained by maximizing their concentration in the respective regions of phase-space. We present applications in time-frequency, wavelet and Gabor analysis.

1. INTRODUCTION

A time-frequency representation of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is a function defined on $\mathbb{R}^d \times \mathbb{R}^d$ whose value at $z = (x, \xi)$ represents the influence of the frequency ξ near x . The short-time Fourier transform (STFT) is a standard choice for such a representation, popular in analysis and signal processing. It is defined, by means of an adequate smooth and fast-decaying window function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, as

$$\mathcal{V}_\varphi f(z) = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t-x)} e^{-2\pi i \xi t} dt, \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

The distribution f can be re-synthesized from its time-frequency content by,

$$(1) \quad f(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{V}_\varphi f(x, \xi) \varphi(t-x) e^{2\pi i \xi t} dx d\xi.$$

This representation is extremely redundant. One of the aims of time-frequency analysis is to provide a representation of an arbitrary signal as a linear combination of elementary time-frequency atoms, which form a less redundant dictionary. The standard choice is to let these atoms be time-frequency shifts of a single window function φ , thus providing a uniform partition of the time-frequency plane. The resulting systems of atoms are known as Gabor frames. However, in certain applications atomic decompositions adapted to a less regular pattern may be required (see for example [29, 4, 8, 13, 42]).

For example, a time-frequency partition may be derived from perceptual considerations. For audio signals, this means that low frequency bins are given a finer resolution than bins in high regions, where better time-resolution is usually desirable, cp. [44, 48]. Such a partition is schematically depicted in the left plot of Figure 1.

More irregular partitions may be desirable whenever the frequency characteristics of an analyzed signal change over time and requires adaptation in both time *and* frequency. For example, adaptive partitions obtained from information theoretic criteria were suggested in [36, 38]. In such a situation, partitions as irregular as shown in the right plot of Figure 1 can be appropriate.

In this article we consider the following problem. Given a - possibly irregular - cover of the time-frequency plane \mathbb{R}^{2d} , we wish to construct a frame for $L^2(\mathbb{R}^d)$ with atoms whose time-frequency concentration follows the shape of the cover members. This allows to vary the trade-off between time and frequency resolution along the time-frequency plane. The adapted frames are constructed by selecting, for each member of a given cover, a family of functions maximizing their concentration in the corresponding region of the time-frequency domain, or phase-space. These functions can be obtained as eigenfunctions of the so-called time-frequency localization operators.

Date: July 24, 2012.

2010 Mathematics Subject Classification. 42C15, 42C40, 41A30, 41A58, 40H05, 47L15.

Key words and phrases. Phase-space, localization operator, frame, short-time Fourier transform, time-frequency analysis, time-scale analysis.

Monika Dörfler was supported by the Austrian Science Fund (FWF):[T384-N13] *Locatif* and by the WWTF project *Audiominer* (MA09-24).

José Luis Romero was supported by the Austrian Science Fund (FWF):[P22746-N13]. He also gratefully acknowledges support from the Austrian Science Fund (FWF): [T384-N13].

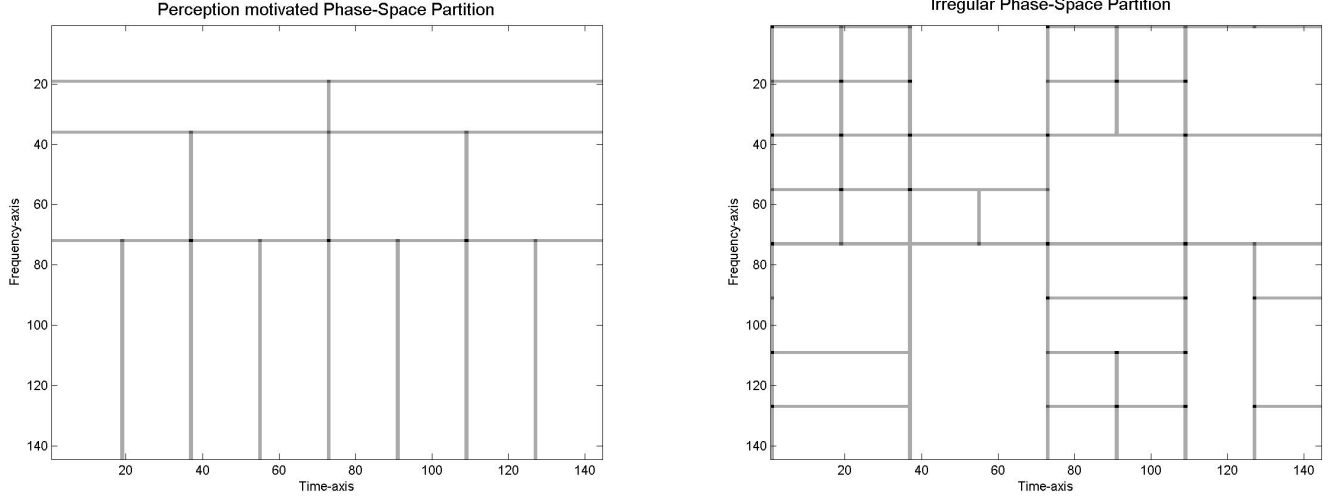


FIGURE 1. Partitions in Time-Frequency

Given a compact set $\Omega \subseteq \mathbb{R}^{2d}$ in the time-frequency plane, the *time-frequency localization operator* H_Ω is defined by masking the coefficients in (1), cf. [14, 15], i.e.

$$(2) \quad H_\Omega f(t) = \int_\Omega \mathcal{V}_\varphi f(x, \xi) \varphi(t - x) e^{2\pi i \xi t} dx dw.$$

H_Ω is self-adjoint and trace-class, so we can consider its spectral decomposition

$$H_\Omega f = \sum_{k=1}^{\infty} \lambda_k \langle f, \phi_k^\Omega \rangle \phi_k^\Omega.$$

The first eigenfunction, ϕ_1^Ω , is optimally concentrated inside Ω in the following sense,

$$\int_\Omega |V_\varphi \phi_1^\Omega(z)|^2 dz = \max_{\|f\|_2=1} \int_\Omega |V_\varphi f(z)|^2 dz.$$

More generally, the first N eigenfunctions of H_Ω form the orthonormal set in $L^2(\mathbb{R}^d)$ that maximizes the quantity $\sum_{j=1}^N \int_\Omega |V_\varphi \phi_j^\Omega(z)|^2 dz$ among all orthonormal sets of N functions in $L^2(\mathbb{R}^d)$. In this sense, their time-frequency profile is optimally adapted to Ω . Figure 2 illustrates this principle by showing some time-frequency boxes Ω along with the STFT and real part of the corresponding localization operator's first eigenfunctions.

In order to construct a frame adapted to a given cover $\{\Omega_\gamma : \gamma \in \Gamma\}$ of \mathbb{R}^{2d} , we select, for each region Ω_γ , the first N_γ eigenfunctions $\psi_{\Omega_\gamma}^1, \dots, \psi_{\Omega_\gamma}^{N_\gamma}$ of the operator H_{Ω_γ} . We will prove that, for $N_\gamma \approx |\Omega_\gamma|$, we obtain a collection of atoms that covers the whole time-frequency plane. Note that this choice of N_γ agrees with the uncertainty principle which roughly says that for each time-frequency region Ω_γ there are only $\approx |\Omega_\gamma|$ degrees of freedom. We allow for covers that are arbitrary in shape as long as they satisfy a mild admissibility condition,

$$(3) \quad B_r(\gamma) \subseteq \Omega_\gamma \subseteq B_R(\gamma), \quad \text{with } \Gamma \text{ a lattice and } R \gg r > 0.$$

Under these conditions we prove the following.

Theorem 1. *Let $\{\Omega_\gamma : \gamma \in \Gamma\}$ be an admissible cover of \mathbb{R}^{2d} . Then, there exists a constant $C > 0$ such that for every choice of N_γ , $C |\Omega_\gamma| \leq N_\gamma \leq N < \infty$, the family of functions $\{\phi_k^{\Omega_\gamma} : \gamma \in \Gamma, 1 \leq k \leq N_\gamma\}$ is a frame of $L^2(\mathbb{R}^d)$. That is, for some constants $0 < A \leq B < +\infty$, the following frame inequality*

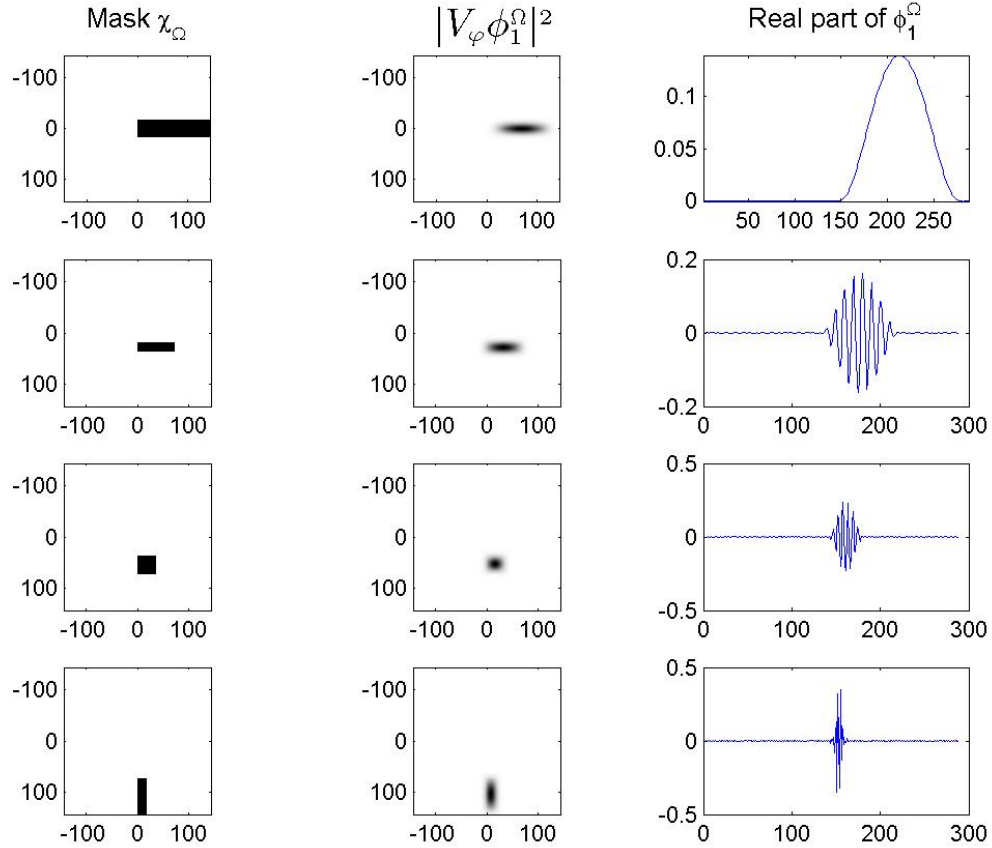


FIGURE 2. Four different rectangular masks in time-frequency domain and the first eigenfunctions of the corresponding localization operators. Middle plots show the absolute value squared of the STFT and right plots show the real part.

holds,

$$A\|f\|_2^2 \leq \sum_{\gamma} \sum_{k=1}^{N_{\gamma}} \left| \langle f, \phi_k^{\Omega_{\gamma}} \rangle \right|^2 \leq B\|f\|_2^2, \quad (f \in L^2(\mathbb{R}^d)).$$

While Theorem 1 was our main motivation, it is just a sample of our results. We work with an abstract model for phase space that allows for a variety of settings. We obtain a complete analogue of Theorem 1 in the context of time-scale analysis where the atoms we design follow a pattern prescribed in the time-scale plane. This flexibility is also further exploited in the context of time-frequency analysis, yielding a variation of Theorem 1 where continuous time-frequency representations are replaced by discrete ones.

1.1. Technical overview. We now give a technical overview, in order to highlight the main steps leading to the proof of Theorem 1 and corresponding statements in Theorem 4, Theorem 5, Theorem 8 and Theorem 9.

The proofs of the main results in this paper are based on two major observations. Firstly, the norm equivalence

$$(4) \quad \|f\|_2^2 \approx \sum_{\gamma} \|H_{\eta_{\gamma}} f\|_2^2,$$

holds for a family of time-frequency localization operators,

$$(5) \quad H_{\eta_\gamma} f(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta_\gamma(x, \xi) \mathcal{V}_\varphi f(x, \xi) \varphi(t - x) e^{2\pi i \xi t} dx dw,$$

provided that the symbols $\eta_\gamma : \mathbb{R}^{2d} \rightarrow [0, +\infty)$ satisfy

$$(6) \quad \sum_{\gamma} \eta_\gamma(z) \approx 1$$

and the enveloping condition

$$(7) \quad \eta_\gamma(z) \leq g(z - \gamma), \text{ for some } g \in L^1(\mathbb{R}^{2d}) \text{ and } \gamma \in \Gamma, \text{ with } \Gamma \subseteq \mathbb{R}^{2d} \text{ a lattice.}$$

The inequalities (4) were first proved in [17] for symbols of the form $\eta_\gamma(z) = h(z - \gamma)$ and $\Gamma = \mathbb{Z}^{2d}$, then for a general lattice in [18], and finally for fully irregular symbols satisfying (7) in [43]. It is interesting to note that the proofs in [17, 18] are *based* on the observation that under condition (6) the norm-equivalence (4) is equivalent to the fact that finitely many eigenfunctions of the operator H_h generate a multi-window Gabor frame over the lattice Γ . The proof of the general case in [43] does not explicitly involve the eigenfunctions of the operators H_{η_γ} nor does it rely on tools specific to Gabor frames on lattices. Thus the question arises whether it is also possible in the irregular case to construct a frame consisting of eigenfunctions of the operators H_{η_γ} . Here, this question is given a positive answer.

Secondly, the observation that (4) remains valid when the operators H_{η_γ} are replaced by finite rank approximations $H_{\eta_\gamma}^\varepsilon$ obtained by thresholding their eigenvalues, cf. Theorem 4, is the core of the proof of our main results. This finite rank approximation is in turn achieved by proving that the operators H_{η_γ} behave “globally” like projectors. More precisely, in Proposition 6 we obtain the following extension of (4)

$$(8) \quad \|f\|_2^2 \approx \sum_{\gamma} \|H_{\eta_\gamma} f\|_2^2 \approx \sum_{\gamma} \|(H_{\eta_\gamma})^2 f\|_2^2.$$

This will allow us to “localize in phase-space” the L^2 -norm estimates relating $H_{\eta_\gamma}^\varepsilon$ and H_{η_γ} .

Note that, in general, the operators H_{η_γ} have infinite rank even if η_γ is the characteristic function of a compact set (see Lemma 1). Consequently $\|H_{\eta_\gamma} f\|_2 \not\approx \|(H_{\eta_\gamma})^2 f\|_2$ and therefore the global properties of the family $\{(H_{\eta_\gamma})^2 : \gamma \in \Gamma\}$ are crucial to prove (4). While the squared operators $(H_{\eta_\gamma})^2$ are not time-frequency localization operators, their time-frequency localizing behavior is preserved under conditions as given by (7): in Section 3 we introduce the notion of a family of operators that is *well-spread in the time-frequency plane* and exploit the fact that the tools from [43] are valid for these operator families.

For clarity, we choose to accentuate the case of time-frequency analysis but all the proofs are carried out in an abstract setting that yields, for example, analogous consequences in time-scale analysis.

1.2. Organization. The article is organized as follows. Section 2 introduces the abstract model for phase-space and Section 3 presents certain key technical notions, in particular the properties required for a family of localization operators to exhibit an almost-orthogonality property. In Section 4, we first prove our results in the context of time-frequency analysis, where some technical problems of the abstract setting do not arise. In addition, in the context of time-frequency analysis, we are able to extend the result on phase-space adapted frames from $L^2(\mathbb{R}^d)$ to an entire class of Banach spaces, the modulation spaces, by exploiting spectral invariance results for pseudo-differential operators. Theorem 5 in Section 4.1 is the general version of Theorem 1 stated above. Section 5 develops the results in the abstract setting. These are then applied to time-scale analysis in Section 5.1. Finally, Section 5.2 contains an additional application of the abstract results to time-frequency analysis, this time using Gabor multipliers, which are time-frequency masking operators related to a discrete time-frequency representation (Gabor frame). The atoms thus obtained maximize their time-frequency concentration with respect to a weight on a discrete time-frequency grid and the resulting frames are relevant in numerical applications.

For clarity, the presentation of the results highlights the case of time-frequency analysis, which was our main motivation. Most of the technicalities in Section 2 are irrelevant to that setting (although they are relevant for time-scale analysis). The reader interested mainly in time-frequency analysis is encouraged to jump directly to Section 4 and then go back to Sections 2 and 3 having a clear example in mind.

2. ABSTRACT PHASE-SPACE

We now introduce a model for *phase-space* that allows us to consider representations such as the STFT and the wavelet transform from a unified point of view. This approach is in the spirit of coorbit theory, cf. [23, 24]. In fact, both the STFT and the wavelet transform may be interpreted as representation coefficients of certain unitary continuous representations of a locally compact group \mathcal{G} . The abstract setting goes without explicit reference to an integral transform.

2.1. Locally compact groups and function spaces. Throughout the article \mathcal{G} will be a locally compact, σ -compact, topological group. The left Haar measure of a set $X \subseteq \mathcal{G}$ will be denoted by $|X|$. Integration will always be considered with respect to the left Haar measure. For $x \in \mathcal{G}$, we denote by L_x and R_x the operators of left and right translation, defined by $L_x f(y) = f(x^{-1}y)$ and $R_x f(y) = f(yx)$. We also consider the involution $f^\vee(x) = f(x^{-1})$.

Given two non-negative functions f, g we write $f \lesssim g$ if there exists a constant $C \geq 0$ such that $f \leq Cg$. We say that $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$. The characteristic function of a set A will be denoted by χ_A .

A set $\Lambda \subseteq \mathcal{G}$ is called *relatively separated* if for some (or any) $V \subseteq \mathcal{G}$, relatively compact neighborhood of the identity, the quantity - called the *spreadness* of Λ -

$$(9) \quad \rho(\Gamma) = \rho_V(\Gamma) := \sup_{x \in \mathcal{G}} \#(\Gamma \cap xV)$$

is finite, i.e. if the amount of elements of Γ that lie in any left translate of V is uniformly bounded.

In generalization of L^p -spaces, we will consider *solid, translation invariant, Banach function (BF) spaces* E , i.e. Banach spaces that satisfy the following.

- (i) E is continuously embedded into $L^1_{\text{loc}}(\mathcal{G})$, the space of complex-valued locally integrable functions on \mathcal{G} .
- (ii) Whenever $f \in E$ and $g : \mathcal{G} \rightarrow \mathbb{C}$ is a measurable function such that $|g(x)| \leq |f(x)|$ a.e., it is true that $g \in E$ and $\|g\|_E \leq \|f\|_E$.
- (iii) E is closed under left and right translations (i.e. $L_x E \subseteq E$ and $R_x E \subseteq E$, for all $x \in \mathcal{G}$) and the following relations hold with the corresponding norm estimates,

$$(10) \quad L^1_u(\mathcal{G}) * E \subseteq E \text{ and } E * L^1_v \subseteq E,$$

where $u(x) := \|L_x\|_{E \rightarrow E}$, $v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{E \rightarrow E}$ and Δ is the modular function of \mathcal{G} .

For each space E we consider weight functions $w : \mathcal{G} \rightarrow (0, +\infty)$ that satisfy the following admissibility conditions.

$$(11) \quad w(x) = \Delta(x^{-1})w(x^{-1}),$$

$$(12) \quad w(xy) \leq w(x)w(y) \text{ (submultiplicativity),}$$

$$(13) \quad w(x) \geq C_{E,w} \max \{u(x), u(x^{-1}), v(x), \Delta(x^{-1})v(x^{-1})\},$$

for some constant $C_{E,w} > 0$. Under these conditions, we say that w is *admissible* for E . It then follows that $w(x) \gtrsim 1$, $L^1_w * E \subseteq E$ and $E * L^1_w \subseteq E$, with the corresponding norm estimates. Moreover, the constants in those estimates depend only on $C_{E,w}$, cf. [23].

If E is a solid translation invariant BF space and $\Gamma \subseteq \mathcal{G}$ is a relatively separated set, we construct discrete versions of E as follows. Fix V , a symmetric relatively compact neighborhood of the identity and let,

$$E_d = E_d(\Gamma) := \left\{ c \in \mathbb{C}^\Gamma \mid \sum_{\lambda} |c_\gamma| \chi_{\gamma V} \in E \right\},$$

and endow it with the norm,

$$\|(c_\gamma)_{\gamma \in \Gamma}\|_{E_d} := \left\| \sum_{\gamma} |c_\gamma| \chi_{\gamma V} \right\|_E.$$

The definition, of course, depends on V but a different choice of V yields the same space with an equivalent norm (this is a consequence of the right invariance of E , see for example [41, Lemma 2.2]). When $E = L_w^p$, for an admissible weight w , the corresponding discrete space $E_d(\Gamma)$ is $\ell_w^p(\Gamma)$, where the weight w is restricted to the set Γ . This follows from the admissibility of w since for $x \in \gamma V$, $w(x) \approx w(\gamma)$.

For a solid, translation invariant BF space E we define the left *Wiener amalgam space* in the following way. We select again a symmetric relatively compact neighborhood of the identity V . For a locally bounded function $f : \mathcal{G} \rightarrow \mathbb{C}$ consider the left *local maximum function* defined by,

$$f^\#(x) := \sup_{y \in V} |f(xy)| = \|f \cdot (L_x \chi_V)\|_\infty, \quad (x \in \mathcal{G}).$$

The Wiener amalgam space is defined as,

$$W(L^\infty, E)(\mathcal{G}) := \{ f \in L_{loc}^\infty(\mathcal{G}) \mid f^\# \in E \},$$

and given the norm $\|f\|_{W(L^\infty, E)} := \|f^\#\|_E$. A different choice for V yields the same space with an equivalent norm (see for example [20, Theorem 1]). The right Wiener amalgam space $W_R(L^\infty, E)$ is defined similarly, this time using the right local maximum function,

$$f_\#(x) := \sup_{y \in V} |f(yx)| = \|f \cdot (R_x \chi_V)\|_\infty,$$

and given the norm $\|f\|_{W_R(L^\infty, E)} := \|f_\#\|_E$. When E is a weighted L^p space, the corresponding amalgam space coincides with the classical $L^\infty - \ell^p$ amalgam space [35, 28]. For the general theory of amalgam spaces in the broader context of possibly non-solid spaces see [20, 21]. In the present article we will be mainly interested in the spaces $W(L^\infty, L_w^1)$ and $W_R(L^\infty, L_w^1)$. We will need the following fact.

Proposition 1. *The spaces $W(L^\infty, L_w^1)$ and $W_R(L^\infty, L_w^1)$ are convolution algebras. That is, the relations $W(L^\infty, L_w^1) * W(L^\infty, L_w^1) \hookrightarrow W(L^\infty, L_w^1)$ and $W_R(L^\infty, L_w^1) * W_R(L^\infty, L_w^1) \hookrightarrow W_R(L^\infty, L_w^1)$ hold together with the corresponding norm estimates.*

Proof. The left amalgam space satisfies the (translation invariance) relation $L_w^1 * W(L^\infty, L_w^1) \hookrightarrow W(L^\infty, L_w^1)$. This is a particular case of [24, Theorem 7.1] and can also be readily deduced from the definitions. Since $W(L^\infty, L_w^1) \hookrightarrow L_w^1$, the statement about $W(L^\infty, L_w^1)$ follows. The involution $f^\vee(x) = f(x^{-1})$ maps $W_R(L^\infty, L_w^1)$ isometrically onto $W(L^\infty, L_w^1)$ (because $V = V^{-1}$) and satisfies $(f * g)^\vee = g^\vee * f^\vee$. Hence the statement about the right amalgam space follows from the one about the left one. \square

2.2. The model for phase-space. In the abstract model for phase-space we consider a solid BF space E (called the environment) and a certain distinguished subspace S_E , which is the range of an idempotent operator P . In the applications S_E will be taken to be the range of an integral transform, such as the STFT in the case of time-frequency analysis. The precise form of the model is taken from [43] and is designed to fit the theory in [23] and [27] (see also [39]).

We list a number of ingredients in the form of two assumptions: (A1) and (A2).

- (A1) – E is a solid, translation invariant BF space, called *the environment*.
- w is an admissible weight for E .
- S_E is a closed complemented subspace of E , called *the atomic subspace*.
- Each function in S_E is continuous.

The second assumption is that the retraction $E \rightarrow S_E$ is given by an operator dominated by right convolution with a kernel in $W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$. In the case of time-frequency or time-scale analysis, the retraction operator is given by the reproducing kernel (see Sections 4 and 5.1).

- (A2) We have an operator P and a non-negative function K satisfying the following.

- $P : W(L^1, L_{1/w}^\infty) \rightarrow L_{1/w}^\infty$ is a (bounded) linear operator,

- $P(E) = S_E$ and $P(f) = f$, for all $f \in S_E$,
- $K \in W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$,
- For $f \in W(L^1, L_{1/w}^\infty)$,

$$(14) \quad |P(f)(x)| \leq \int_{\mathcal{G}} |f(y)| K(y^{-1}x) dy, \quad (x \in \mathcal{G}).$$

When $E = L^2(\mathcal{G})$ we will also assume the following,

(A3) $P : L^2(\mathcal{G}) \rightarrow S_{L^2}$ is the orthogonal projection.

For the remainder of Section 2 we assume (A1) and (A2). Under these conditions the following holds.

Proposition 2. [43, Prop. 3]

- (a) P boundedly maps E into $W(L^\infty, E)$.
- (b) $S_E \hookrightarrow W(L^\infty, E)$.
- (c) If $f \in W(L^1, L_{1/w}^\infty)$, then $\|P(f)\|_{L_{1/w}^\infty} \lesssim \|f\|_{W(L^1, L_{1/w}^\infty)} \|K\|_{W_R(L^\infty, L_w^1)}$.
- (d) If $f \in W(L^1, L^\infty)$, then $\|P(f)\|_{L^\infty} \lesssim \|f\|_{W(L^1, L^\infty)} \|K\|_{W_R(L^\infty, L_w^1)}$.

Remark 1. Since $w \gtrsim 1$, $L^\infty \hookrightarrow L_{1/w}^\infty$.

Remark 2. The estimates in Prop. 2 hold uniformly for all the spaces E with the same weight w and the same constant $C_{E,w}$ (cf. (13)). This is relevant to the applications, where the same projection P is used with different spaces E and corresponding subspaces $S_E = P(E)$.

2.3. Phase-space multipliers. Time-frequency localization operators (2) are a particular example, be it of overwhelming practical importance, cf. [14, 16, 49, 50, 11], of the general concept of *phase-space multipliers*.

For $m \in L^\infty(\mathcal{G})$, the phase-space multiplier $M_m : S_E \rightarrow S_E$ with *symbol* m is defined by,

$$(15) \quad M_m(f) := P(mf), \quad (f \in S_E).$$

The operator M_m is clearly bounded by Proposition 2 and the solidity of E ,

$$(16) \quad \|M_m(f)\|_E \lesssim \|m\|_\infty \|f\|_E, \quad (f \in S_E).$$

If S_E is the range of the STFT (and P is the orthogonal projection onto it) the corresponding operators are unitarily equivalent to the time-frequency localization operators ([14, 11, 6, 12]). More generally, if the space S_E is the range of the abstract wavelet transform associated with a unitary representation of \mathcal{G} , these operators are called localization operators or wavelet multipliers (see for example [50, 37]).

Remark 3. In this article we will mainly be concerned with multipliers with bounded symbols m . However, due to the regularizing effect of P , the condition that m be bounded is by no means necessary for $M_m : S_E \rightarrow S_E$ to be bounded. See for example [25, 11] for sharper boundedness results for time-frequency localization operators.

For future reference we note some Hilbert-space properties of phase-space multipliers (when $E = L^2(\mathcal{G})$).

Proposition 3. Let $E = L^2(\mathcal{G})$ and assume (A1), (A2) and (A3). Then, the following hold.

- (a) Let $m_1, m_2 \in L^\infty(\mathcal{G})$ be real-valued. If $m_1 \leq m_2$ a.e., then $M_{m_1} \leq M_{m_2}$ as operators. In particular if m is non-negative and bounded, then M_m is a positive operator.
- (b) Let $m \in L^1(\mathcal{G}) \cap L^\infty(\mathcal{G})$ be non-negative. Then $M_m : S_{L^2} \rightarrow S_{L^2}$ is trace-class and $\text{trace}(M_m) \lesssim \|m\|_1$.

Proof. To prove (a), let $f \in S_{L^2}$ and note that by (A3),

$$\begin{aligned} \langle M_{m_1} f, f \rangle &= \langle P(m_1 f), f \rangle = \langle m_1 f, f \rangle \\ &= \int_{\mathcal{G}} m_1(x) |f(x)|^2 dx \\ &\leq \int_{\mathcal{G}} m_2(x) |f(x)|^2 dx = \langle M_{m_2} f, f \rangle. \end{aligned}$$

Let us now prove (b). For $f \in S_{L^2}$ and $x \in \mathcal{G}$, by (14)

$$|f(x)| \leq \int_{\mathcal{G}} |f(y)| K(y^{-1}x) dy = \int_{\mathcal{G}} |f(y)| K^\vee(x^{-1}y) dy \leq \|f\|_2 \|K^\vee\|_2.$$

This is finite because $K^\vee \in W_R(L^\infty, L_w^1) \subseteq L^2$. Hence $f(x) = \langle f, E_x \rangle$, for some function $E_x \in S_{L^2}$ with $\|E_x\|_2 \leq \|K^\vee\|_2$.

Let $\{e_k\}_k$ be an orthonormal basis of S_{L^2} . Since by (a) M_m is a positive operator, it suffices to check that $\sum_k \langle M_m e_k, e_k \rangle \lesssim \|m\|_1$ (see for example, [45, Theorem 2.14]). To this end, note that for $x \in \mathcal{G}$,

$$\sum_k |e_k(x)|^2 = \sum_k |\langle e_k, E_x \rangle|^2 = \|E_x\|_2^2 \leq \|K^\vee\|_2^2.$$

Hence,

$$\sum_k \langle M_m e_k, e_k \rangle = \sum_k \int_{\mathcal{G}} m(x) |e_k(x)|^2 \leq \|K^\vee\|_2^2 \|m\|_1.$$

This completes the proof. \square

3. WELL-SPREAD FAMILIES OF OPERATORS

One of the main technical insights of this article is the fact, that some important tools used in the investigation of families phase-space multipliers only depend on certain phase-space localization estimates. To formalize this observation, we now introduce the key concept of families of operators that are well-spread in phase-space. These families of operators are required to be dominated by product-convolution operators $f \mapsto (fg(\gamma^{-1}\cdot)) * \Theta$ centered at suitably distributed nodes γ . The advantage of working with well-spread families instead of just families of phase-space multipliers is that well-spreadness is stable under various operations, e.g. finite composition. This will be essential to the proofs of our main results.

In this section, we assume that (A1) and (A2) from Section 2.2 hold.

Definition 1. Let $\Gamma \subseteq \mathcal{G}$ be a relatively separated set, $\Theta \in W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$ and $g \in W_R(L^\infty, L_w^1)(\mathcal{G})$ be non-negative functions. A family of operators $\{T_\gamma : S_E \rightarrow S_E : \gamma \in \Gamma\}$ is called well-spread with envelope (Γ, Θ, g) if the following pointwise estimate holds

$$(17) \quad |T_\gamma f(x)| \leq \int_{\mathcal{G}} g(\gamma^{-1}y) |f(y)| \Theta(y^{-1}x) dy, \quad (\gamma \in \Gamma, x \in \mathcal{G}).$$

When we do not want to emphasize the role of the envelope, we simply say that $\{T_\gamma\}_\gamma$ is a well-spread family of operators; this implies the existence of an adequate envelope.

The canonical example of a well-spread family of operators is a family of phase-space multipliers $\{M_{\eta_\gamma} \mid \gamma \in \Gamma\}$ associated with an adequate family of symbols. A family of symbols $\{\eta_\gamma : \mathcal{G} \rightarrow \mathbb{C} \mid \gamma \in \Gamma\}$ is called *well-spread* if it satisfies the following.

- $\Gamma \subseteq \mathcal{G}$ is a relatively separated set.
- There is a function $g \in W_R(L^\infty, L_w^1)$ such that

$$(18) \quad |\eta_\gamma(x)| \leq g(\gamma^{-1}x), \quad (x \in \mathcal{G}, \gamma \in \Gamma).$$

The pair (Γ, g) is called an envelope for $\{\eta_\gamma\}_\gamma$. Together with the properties of the convolution kernel K dominating the projection onto S_E , we immediately obtain a family of well-spread operators.

Proposition 4. Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of symbols. Then, the corresponding family of phase-space multipliers $\{M_{\eta_\gamma}\}_\gamma$ is well-spread.

Proof. It follows readily from the definitions that if (Γ, g) is an envelope for $\{\eta_\gamma\}_\gamma$, then (Γ, K, g) is an envelope for $\{M_{\eta_\gamma}\}_\gamma$. \square

The reason why we introduce the concept of well-spread families of operators is that composition of phase-space multipliers usually fails to yield a phase-space multiplier. However, the estimate in (17) is stable under various operations. In this article we will be mainly interested in the following.

Proposition 5. *Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of symbols. Then, the family of operators $\{(M_{\eta_\gamma})^2 : \gamma \in \Gamma\}$ is well-spread.*

Proof. If (Γ, g) is an envelope for $\{\eta_\gamma\}_\gamma$ then,

$$\begin{aligned} |(M_{\eta_\gamma})^2 f(x)| &\leq \int_{\mathcal{G}} g(\gamma^{-1}y) |M_{\eta_\gamma} f(y)| K(y^{-1}x) dy \\ &\leq \int_{\mathcal{G}} \int_{\mathcal{G}} g(\gamma^{-1}y) g(\gamma^{-1}z) |f(z)| K(z^{-1}y) K(y^{-1}x) dy dz \\ &\leq \|g\|_\infty \int_{\mathcal{G}} g(\gamma^{-1}z) |f(z)| (K * K)(z^{-1}x) dz. \end{aligned}$$

Hence, if we set $\Theta := K * K$, it follows that $(\Gamma, \Theta, \|g\|_\infty g)$ is an envelope for $\{(M_{\eta_\gamma})^2\}_\gamma$. Since K belongs to $W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$, by Proposition 1, so does Θ . \square

3.1. Almost-orthogonality estimates. We now introduce one of the key estimates of the article.

Theorem 2 ([43]). *Let $\{T_\gamma : \gamma \in \Gamma\}$ be a well-spread family of operators. Suppose that the operator $\sum_\gamma T_\gamma : S_E \rightarrow S_E$ is invertible. Then, the following norm equivalence holds,*

$$(19) \quad \|(\|T_\gamma f\|_{L^2(\mathcal{G})})_{\gamma \in \Gamma}\|_{E_d} \approx \|f\|_E, \quad (f \in S_E).$$

Remark 4. *The implicit constants depend only on $\|K\|_{W(L^\infty, L_w^1)}$, $\|K\|_{W_R(L^\infty, L_w^1)}$, $\|\Theta\|_{W(L^\infty, L_w^1)}$, $\|\Theta\|_{W_R(L^\infty, L_w^1)}$, $\|g\|_{W_R(L^\infty, L_w^1)}$, $\rho(\Gamma)$ (cf. (9)) and $C_{E,w}$ (cf. (13)).*

Theorem 2 is proved in [43] for families of phase-space multipliers associated with well-spread families of symbols. However, the argument works without changes for the case of general well-spread families of operators. Indeed, the definition in (17) is tailored to the requirements of the proof in [43]. For completeness, we sketch a proof of Theorem 2 in the Appendix.

We also point out that the L^2 -norm in (19) can be replaced by any solid translation invariant norm (cf. Assumption (B1) in [43]). However, the by far most important case is the one of L^2 . Indeed the core of our argument in this article makes use of (19) to extrapolate certain thresholding estimates from L^2 to other Banach spaces.

We regard Theorem 2 as an almost orthogonality principle. Its main insight is that, because of the phase-space localization of the family $\{T_\gamma\}_\gamma$, the effect of each individual operator within the sum $\sum_\gamma T_\gamma$ decouples.

Corollary 1. *Let $\{T_\gamma : \gamma \in \Gamma\}$ be a well-spread family of operators. Suppose that $E = L^2(\mathcal{G})$ and that the operator $\sum_\gamma T_\gamma : S_{L^2} \rightarrow S_{L^2}$ is invertible. Then, so is the operator $\sum_\gamma T_\gamma^* T_\gamma : S_{L^2} \rightarrow S_{L^2}$.*

Proof. By Theorem 2, the invertibility of $\sum_\gamma T_\gamma$ implies that for $f \in S_{L^2}$,

$$\|f\|_2^2 \approx \sum_\gamma \|T_\gamma f\|_2^2 = \left\langle \sum_\gamma T_\gamma^* T_\gamma f, f \right\rangle.$$

Hence $\sum_\gamma T_\gamma^* T_\gamma$ is positive definite and therefore invertible on S_{L^2} . \square

Remark 5. *In the context of time-frequency analysis, Corollary 1 will be extended to other spaces E besides $L^2(\mathcal{G})$.*

Remark 6. *Corollary 1 gives further support to the interpretation of Theorem 2 as an almost orthogonality principle. Let us consider for simplicity a well-spread family of self-adjoint operators $\{T_\gamma\}_\gamma$. Since $(\sum_\gamma T_\gamma)^2 = \sum_\gamma (T_\gamma)^2 + \sum_{\gamma, \gamma' / \gamma \neq \gamma'} T_\gamma T_{\gamma'}$ and $(\sum_\gamma T_\gamma)^2$ is invertible if and only if $\sum_\gamma T_\gamma$ is, Corollary 1 says that the invertibility of $(\sum_\gamma T_\gamma)^2$ implies that of its “diagonal part” $\sum_\gamma (T_\gamma)^2$.*

4. THE CASE OF TIME-FREQUENCY ANALYSIS

In this section we consider the case $\mathcal{G} = \mathbb{R}^{2d}$, and we let the phase-space be the time-frequency (TF) plane. In time-frequency analysis, function spaces defined via their members' properties in phase space are known as *modulation spaces*, whose definition we give next. Let $w : \mathbb{R}^{2d} \rightarrow (0, +\infty)$ be a submultiplicative, even weight that satisfies the GRS condition,

$$\lim_{n \rightarrow \infty} w(nx)^{1/n} = 1, \quad (x \in \mathbb{R}^{2d}).$$

Let $v : \mathbb{R}^{2d} \rightarrow (0, +\infty)$ be a w -moderated weight; that is

$$(20) \quad v(x+y) \leq C_v w(x)v(y), \quad (x, y \in \mathbb{R}^{2d}),$$

for some constant $C_v > 0$. Assume further that v is moderated by a polynomial weight¹. Given a non-zero Schwartz-class function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the modulation space $M_v^p(\mathbb{R}^d)$, ($1 \leq p \leq +\infty$) is defined as the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\varphi f$ belongs to the weighted Lebesgue space $L_v^p(\mathbb{R}^{2d})$. The space M_v^p is given the norm $\|f\|_{M_v^p} := \|V_\varphi f\|_{L_v^p}$. A different choice for φ yields the same space with an equivalent norm. Moreover, not only Schwartz-class functions can be used as windows; any non-zero $\varphi \in M_w^1$ is adequate, and we will make this assumption hereafter. For convenience we also assume that $\|\varphi\|_2 = 1$. When $p = 2$ and $v \equiv 1$, M_v^p is just $L^2(\mathbb{R}^d)$.

We now indicate how the abstract setting from Section 2 applies to time-frequency analysis. We let $\mathcal{G} = \mathbb{R}^{2d}$, $E = L_v^p(\mathbb{R}^{2d})$ and $S_E = S_v^p := V_\varphi(L_v^p)$. The projection $P : L_v^p(\mathbb{R}^{2d}) \rightarrow S_v^p$ is given by $P(f) := V_\varphi V_\varphi^*(f)$. If $E = L^2(\mathbb{R}^{2d})$, P is in fact the orthogonal projector onto S^2 . The estimate in (14) is satisfied with $K := |V_\varphi(\varphi)|$. By definition, $\varphi \in M_w^1$ means that $V_\varphi(\varphi) \in L_w^1(\mathbb{R}^{2d})$, but it is well-known that in this case, $V_\varphi(\varphi)$ also belongs to $W(L^\infty, L_w^1)(\mathbb{R}^{2d})$ (see [30, Proposition 12.1.11]; note that this fact can also be derived from the norm equivalence in Proposition 2.)

Since $\mathcal{G} = \mathbb{R}^{2d}$ is abelian, $W_R(L^\infty, L_w^1)(\mathbb{R}^{2d}) = W(L^\infty, L_w^1)(\mathbb{R}^{2d})$, and a family of symbols $\{\eta_\gamma : \mathbb{R}^{2d} \rightarrow \mathbb{C} \mid \gamma \in \Gamma\}$ is well-spread if, for a relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$ and $g \in W(L^\infty, L_w^1)(\mathbb{R}^{2d})$

$$|\eta_\gamma(x)| \leq g(x - \gamma), \quad (x \in \mathbb{R}^{2d}, \gamma \in \Gamma).$$

Section 3.²

In the present context of time-frequency analysis, we consider operators in the signal domain, which is unitarily mapped to phase-space by V_φ . Then, in equivalence to the definition given in Section 3, a family of operators $\{H_\gamma : M_{1/w}^\infty(\mathbb{R}^d) \rightarrow M_{1/w}^\infty(\mathbb{R}^d) \mid \gamma \in \Gamma\}$ is said to be *well-spread in the time-frequency plane* if there exists an envelope (Γ, Θ, g) , such that

- $\Gamma \subseteq \mathbb{R}^{2d}$ is relatively separated,
- $\Theta, g \in W(L^\infty, L_w^1)(\mathbb{R}^{2d})$ and the following estimate holds for $x \in \mathbb{R}^{2d}$ and $\gamma \in \Gamma$:

$$(21) \quad |V_\varphi H_\gamma f(x)| \leq \int_{\mathbb{R}^{2d}} |V_\varphi f(y)| \Theta(x - y) g(y - \gamma) dy.$$

Note that $\{H_\gamma\}_\gamma$ is well-spread in the TF plane if and only if the family of operators

$$\{V_\varphi H_\gamma V_\varphi^* : S_{1/w}^\infty \rightarrow S_{1/w}^\infty \mid \gamma \in \Gamma\}$$

is well-spread in the sense of Section 3.

Note also that, if $\{H_\gamma\}_\gamma$ is well-spread in the time-frequency plane, then, because of (21), each operator H_γ maps $M_v^p(\mathbb{R}^d)$ into $M_v^p(\mathbb{R}^d)$ with a norm bound independent of γ ,

$$\|H_\gamma f\|_{M_v^p} \leq C_v \|g\|_\infty \|\Theta\|_{L_w^1} \|f\|_{M_v^p}.$$

¹This assumption is only made in order to define modulation spaces as subsets of the class of tempered distributions. For a general weight, the space M_v^p can still be constructed as an abstract coorbit space.

²We remark that for the results in this section the assumption $g \in W(L^\infty, L_w^1)(\mathbb{R}^{2d})$ can be relaxed to $g \in L^1(\mathbb{R}^{2d})$, but the proofs would be more technical with little practical gain (cf. [43, Section 2.4] and the Appendix).

Remark 7. In parallel to Proposition 4, if $\{\eta_\gamma\}_\gamma$ is a well-spread family with envelope (Γ, g) , then the corresponding family of time-frequency localization operators $\{H_{\eta_\gamma}\}_\gamma$ is well-spread in the time-frequency plane with envelope (Γ, K, g) .

In the case of time-frequency analysis we can strengthen Theorem 2 by means of spectral invariance results for pseudo-differential operators [46, 47, 32, 31]. Due to these results, the invertibility assumption in Theorem 2 can be replaced by assuming invertibility on $L^2(\mathbb{R}^d)$ only.

Theorem 3. Let $\{H_\gamma : \gamma \in \Gamma\}$ be well-spread in the TF plane. Suppose that the operator $\sum_\gamma H_\gamma : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is invertible. Then, also $\sum_\gamma H_\gamma : M_v^p \rightarrow M_v^p$ is invertible and the following norm equivalence holds, for $1 \leq p \leq +\infty$ and w -dominated weights v .

$$\|f\|_{M_v^p} \approx \left(\sum_{\gamma \in \Gamma} \|H_\gamma f\|_{L^2(\mathcal{G})}^p v(\gamma)^p \right)^{1/p}, \quad (f \in M_v^p(\mathbb{R}^d)),$$

with the usual modification for $p = \infty$.

Remark 8. The estimates hold uniformly for $1 \leq p \leq +\infty$ and any family of weights having a uniform constant C_v (cf. (20)).

Proof of Theorem 3. Let $T_\gamma := V_\varphi H_\gamma V_\varphi^* : S_v^p \rightarrow S_v^p$. The family $\{T_\gamma : \gamma \in \Gamma\}$ is well-spread (in the sense of Section 3). In order to apply Theorem 2 we need to show that $\sum_\gamma T_\gamma : S_v^p \rightarrow S_v^p$ is invertible. This is the case if and only if $R_{p,v} := \sum_\gamma H_\gamma : M_v^p \rightarrow M_v^p$ is invertible. By assumption we know that $R_2 : \sum_\gamma H_\gamma : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is invertible.

Let $\{\pi(\lambda)h : \lambda \in \Lambda\}$ be a (Gabor) frame for $L^2(\mathbb{R}^d)$, where $h \in M_w^1$ and $\Lambda \subseteq \mathbb{R}^{2d}$ is a lattice (for example we can take $h(x) = e^{-\pi|x|^2}$ and $\Lambda := \mathbb{Z}^d \times 1/2\mathbb{Z}^d$). Using (21) and the fact that $|V_\varphi \pi(\lambda)h(x)| = |V_\varphi h(x - \lambda)|$ we estimate,

$$\begin{aligned} |\langle R_{p,v} \pi(\lambda)h, \pi(\mu)h \rangle| &= |\langle V_\varphi R_{p,v} \pi(\lambda)h, V_\varphi \pi(\mu)h \rangle| \\ &\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\varphi \pi(\lambda)h(y)| \Theta(x - y) \sum_\gamma g(y - \gamma) |V_\varphi \pi(\mu)h(x)| dy dx \\ &\lesssim \|g\|_{W(L^\infty, L_w^1)} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\varphi h(y - \lambda)| \Theta(x - y) |V_\varphi h(x - \mu)| dy dx \\ &\lesssim (|V_\varphi h| * |V_\varphi h| * \Theta)(\lambda - \mu). \end{aligned}$$

Since $h \in M_w^1(\mathbb{R}^d)$ and $\Theta \in W(L^\infty, L_w^1)$, $|V_\varphi h| * |V_\varphi h| * \Theta \in W(L^\infty, L_w^1)$. Therefore, the restriction of that function to Λ gives an ℓ_w^1 sequence. By [32, Theorem 3.2 and Corollary 3.3], R_2 is a pseudodifferential operator with symbol in the weighted Sjöstrand's class $M_{\tilde{w}}^{\infty,1}$, where $\tilde{w}(z_1, z_2) = (-z_2, z_1)$ and $(z_1, z_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Now the invertibility of $R_{p,v}$ follows from the invertibility of R_2 and [32, Corollary 4.7] (see also [31]). \square

4.1. Frames of eigenfunctions of time-frequency localization operators. We now turn to the construction of frames comprised of eigenfunctions of localization operators. Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of non-negative functions on \mathbb{R}^{2d} and consider the corresponding family of time-frequency localization operators,

$$H_{\eta_\gamma} f = V_\varphi^*(\eta_\gamma V_\varphi f).$$

Since each η_γ is non-negative and belongs to $L^1(\mathbb{R}^{2d})$, H_{η_γ} is positive and trace class, and $\text{trace}(H_{\eta_\gamma}) = \|\eta_\gamma\|_1$ (see [7, 25, 50])³. Hence $H_{\eta_\gamma} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ can be diagonalized as,

$$(22) \quad H_{\eta_\gamma} f = \sum_{k \geq 1} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma, \quad (f \in L^2(\mathbb{R}^d))$$

³Alternatively, use Prop. 3

where $\{\phi_k^\gamma \mid k \in \mathbb{N}\}$ is an orthonormal subset of $L^2(\mathbb{R}^d)$ - possibly incomplete if $\ker(H_{\eta_\gamma}) \neq \{0\}$ - and $(\lambda_k^\gamma)_k$ is a non-increasing sequence of non-negative real numbers satisfying,

$$(23) \quad \sum_k \lambda_k^\gamma = \text{trace}(H_{\eta_\gamma}) = \|\eta_\gamma\|_1.$$

The time-frequency profile of the functions $\{\phi_k^\gamma \mid k \in \mathbb{N}\}$ is optimally adapted to the mask η_γ in the following sense. For each $N \in \mathbb{N}$, the set $\{\phi_1^\gamma, \dots, \phi_N^\gamma\}$ is an orthonormal set maximizing the quantity,

$$\sum_{k=1}^N \int_{\mathbb{R}^{2d}} \eta_\gamma(z) |V_\varphi f_k(z)|^2 dz,$$

among all orthonormal sets $\{f_1, \dots, f_N\} \subseteq L^2(\mathbb{R}^d)$. Moreover, the functions ϕ_k^γ enjoy other time-frequency concentration properties. For example, since $\eta_\gamma \in L_w^1$ and $\varphi \in M_w^1$, it is easy to see that $\phi_k^\gamma \in M_w^1$ (see for example [18, Lemma 5]).

For each $\varepsilon > 0$, we define the operator $H_{\eta_\gamma}^\varepsilon$ by applying a threshold to the eigenvalues of H_{η_γ} ,

$$(24) \quad H_{\eta_\gamma}^\varepsilon f := \sum_{k: \lambda_k^\gamma > \varepsilon} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma, \quad (f \in L^2(\mathbb{R}^d)).$$

Hence,

$$(25) \quad \|H_{\eta_\gamma}^\varepsilon f\|_2 \leq \|H_{\eta_\gamma} f\|_2 \leq \|H_{\eta_\gamma}^\varepsilon f\|_2 + \varepsilon \|f\|_2, \quad (f \in L^2(\mathbb{R}^d)).$$

We now state the main technical result.

Theorem 4. *Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of non-negative symbols on \mathbb{R}^{2d} such that $\sum_\gamma \eta_\gamma \approx 1$. Then, there exist constants $0 < c \leq C < +\infty$ such that for all sufficiently small $\varepsilon > 0$,*

$$c \|f\|_{M_v^p} \leq \left(\sum_{\gamma \in \Gamma} \|H_{\eta_\gamma}^\varepsilon f\|_{L^2(\mathcal{G})}^p v(\gamma)^p \right)^{1/p} \leq C \|f\|_{M_v^p}, \quad (f \in M_v^p(\mathbb{R}^d)),$$

with the usual modification for $p = \infty$.

The choice of ε and the estimates are uniform for $1 \leq p \leq +\infty$ and any family of weights having a uniform constant C_v (cf. (20)).

The strategy to prove Theorem 4 is to show that “globally” the operators $\{H_{\eta_\gamma}\}_\gamma$ behave like projectors. Note that in general $\|(H_{\eta_\gamma})^2 f\|_2 \not\approx \|H_{\eta_\gamma} f\|_2$. Indeed, if η_γ is the characteristic function of a set with non-empty interior, then the range operator H_{η_γ} is infinitely dimensional (see Lemma 1) and $\|(H_{\eta_\gamma})^2 \phi_k^\gamma\|_2 = (\lambda_k^\gamma)^2 \not\approx \lambda_k^\gamma = \|H_{\eta_\gamma} \phi_k^\gamma\|_2$. However, we prove the following.

Proposition 6. *Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of non-negative symbols on \mathbb{R}^{2d} such that $\sum_\gamma \eta_\gamma \approx 1$. Then, for $f \in M_v^p(\mathbb{R}^d)$,*

$$(26) \quad \|f\|_{M_v^p} \approx \left(\sum_{\gamma \in \Gamma} \|H_{\eta_\gamma} f\|_{L^2(\mathcal{G})}^p v(\gamma)^p \right)^{1/p} \approx \left(\sum_{\gamma \in \Gamma} \|(H_{\eta_\gamma})^2 f\|_{L^2(\mathcal{G})}^p v(\gamma)^p \right)^{1/p}.$$

Proof. The estimate $\|f\|_{M_v^p} \approx \|(\|H_{\eta_\gamma} f\|_2)_\gamma\|_{\ell_v^p}$ is contained in [18] for the case of families of symbols consisting of lattice translates of a single function, and in [43] in the present generality. It follows readily from Theorem 3 by the following argument. Since the symbols η_γ satisfy $m := \sum_\gamma \eta_\gamma \geq A$ for some constant $A > 0$ it follows that,

$$\begin{aligned} \left\langle \sum_\gamma H_{\eta_\gamma} f, f \right\rangle &= \langle H_m f, f \rangle = \langle V_\varphi^*(m V_\varphi f), f \rangle \\ &= \langle m V_\varphi f, V_\varphi f \rangle = \int_{\mathbb{R}^{2d}} m(z) |V_\varphi f(z)|^2 dz \geq A \|V_\varphi f\|_2^2 = A \|f\|_2^2. \end{aligned}$$

Hence, $\sum_\gamma H_{\eta_\gamma} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is invertible (cf. [12, Last remark]). Now Prop. 4 and Theorem 3 yield the desired estimate.

For the estimate involving the squared operators, note that Corollary 1 implies that $\sum_{\gamma} (H_{\eta_{\gamma}})^2 : S_{L^2} \rightarrow S_{L^2}$ is also invertible. According to Prop. 5, the family $\{(H_{\eta_{\gamma}})^2\}_{\gamma}$ is also well-spread. Hence, a second application of Theorem 3 concludes the proof. \square

Proof of Theorem 4. Given $\varepsilon > 0$ and $f \in M_v^p(\mathbb{R}^d)$ we apply (25) to $H_{\eta_{\gamma}}(f)$ to obtain

$$\|(H_{\eta_{\gamma}})^2 f\|_2 \leq \|H_{\eta_{\gamma}}^{\varepsilon} H_{\eta_{\gamma}} f\|_2 + \varepsilon \|H_{\eta_{\gamma}} f\|_2.$$

Since $H_{\eta_{\gamma}}^{\varepsilon}$ and $H_{\eta_{\gamma}}$ commute, $\|H_{\eta_{\gamma}}^{\varepsilon} H_{\eta_{\gamma}} f\|_2 = \|H_{\eta_{\gamma}} H_{\eta_{\gamma}}^{\varepsilon} f\|_2 \lesssim \|H_{\eta_{\gamma}}^{\varepsilon} f\|_2$. Hence,

$$\|(H_{\eta_{\gamma}})^2 f\|_2 \lesssim \|H_{\eta_{\gamma}}^{\varepsilon} f\|_2 + \varepsilon \|H_{\eta_{\gamma}} f\|_2.$$

Taking ℓ_v^p norm on γ yields,

$$\|(\|(H_{\eta_{\gamma}})^2 f\|_2)_{\gamma}\|_{\ell_v^p} \lesssim \|(\|H_{\eta_{\gamma}}^{\varepsilon} f\|_2)_{\gamma}\|_{\ell_v^p} + \varepsilon \|(\|H_{\eta_{\gamma}} f\|_2)_{\gamma}\|_{\ell_v^p}.$$

Using the estimates in Prop. 6 we get,

$$\|f\|_{M_v^p} \leq C \|(\|H_{\eta_{\gamma}}^{\varepsilon} f\|_2)_{\gamma}\|_{\ell_v^p} + c\varepsilon \|f\|_{M_v^p},$$

for some constants c, C . Hence,

$$(1 - c\varepsilon) \|f\|_{M_v^p} \leq C \|(\|H_{\eta_{\gamma}}^{\varepsilon} f\|_2)_{\gamma}\|_{\ell_v^p}.$$

This gives the desired lower bound (for all $0 < \varepsilon < 1/c$). The upper bound follows from the first inequality in (25) and Prop. 6. \square

Remark 9. Note that the proof of Theorem 4 only uses the estimate in (25) and the fact that $H_{\eta_{\gamma}}$ and $H_{\eta_{\gamma}}^{\varepsilon}$ commute. Hence, more general thresholding rules besides the one in (24) can be used.

Finally, we obtain the desired result on frames of eigenfunctions.

Theorem 5. Let $\{\eta_{\gamma}\}_{\gamma}$ be a well-spread family of non-negative symbols on \mathbb{R}^{2d} such that $\sum_{\gamma} \eta_{\gamma} \approx 1$ and v a w -moderated weight. Then, there exists a constant $\alpha > 0$ such that, for every choice of finite subsets of eigenfunctions $\{\phi_k^{\gamma} \mid \gamma \in \Gamma, 1 \leq k \leq N_{\gamma}\}$ with $\alpha \|\eta_{\gamma}\|_1 \leq N_{\gamma} \leq N < +\infty$, the following frame estimates hold simultaneously for all $1 \leq p \leq +\infty$, with the usual modification for $p = \infty$:

$$\|f\|_{M_v^p} \approx \left(\sum_{\gamma} \sum_{k=1}^{N_{\gamma}} |\langle f, \lambda_k^{\gamma} \phi_k^{\gamma} \rangle|^p v(\gamma)^p \right)^{1/p}.$$

Moreover, α can be chosen uniformly for any class of weights v having a uniform constant C_v (cf. (20)).

Remark 10. As opposed to other constructions that partition either the time or frequency domain (see e.g. [22, 4, 9]), the symbols η_{γ} partition the time-frequency plane simultaneously.

Remark 11. Note that, if (Γ, Θ, g) is an envelope for $\{\eta_{\gamma}\}_{\gamma}$, since $\|\eta_{\gamma}\|_1 \leq \|g\|_1$, in Theorem 5 it is always possible to make a uniform choice $N_{\gamma} = N_0$.

Remark 12 (Characteristic functions). When η_{γ} is the characteristic function of a set Ω_{γ} , the condition $\sum_{\gamma} \eta_{\gamma} \approx 1$ means that the family of sets $\{\Omega_{\gamma} : \gamma \in \Gamma\}$ is a cover of \mathbb{R}^{2d} having bounded overlaps. The well-spreadness condition requires the sets Ω_{γ} have bounded measure and ‘‘eccentricity’’. For example, the family of characteristic functions is well-spread if for some $R > 0$, $\Omega_{\gamma} \subseteq B_R(\gamma)$, for all γ belonging to a well-spread set Γ (e.g. a lattice).

In Theorem 5, we pick $\approx |\Omega_{\gamma}|$ eigenfunctions from each time-frequency localization operator. This agrees with the uncertainty principle that says that each region of the time-frequency plane Ω_{γ} represents $\approx |\Omega_{\gamma}|$ degrees of freedom in signal space.

Proof of Theorem 5. For every $\gamma \in \Gamma$ and $\varepsilon > 0$, let $I_\gamma^\varepsilon := \{k \in \mathbb{N} / \lambda_k^\gamma > \varepsilon\}$, which is a finite set. Using Theorem 4, Proposition 6 and the orthonormality of the eigenfunctions, we can find a value of $\varepsilon > 0$ such that

$$\begin{aligned} \|f\|_{M_v^p} &\approx \left(\sum_\gamma \left(\sum_{k \in \mathbb{N}} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^2 \right)^{p/2} v(\gamma)^p \right)^{1/p} \\ &\approx \left(\sum_\gamma \left(\sum_{k \in I_\gamma^\varepsilon} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^2 \right)^{p/2} v(\gamma)^p \right)^{1/p}, \end{aligned}$$

which implies that for any choice of subsets of indices $J_\gamma^\varepsilon \supseteq I_\gamma^\varepsilon$ we also have,

$$(27) \quad \|f\|_{M_v^p} \approx \left(\sum_\gamma \left(\sum_{k \in J_\gamma^\varepsilon} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^2 \right)^{p/2} v(\gamma)^p \right)^{1/p}.$$

Furthermore, due to (23) we have $\#I_\gamma^\varepsilon \leq \varepsilon^{-1} \sum_k \lambda_k^\gamma = \varepsilon^{-1} \|\eta_\gamma\|_1$. Hence, setting $\alpha := \varepsilon^{-1}$, we ensure that for $N_\gamma \geq \alpha \|\eta_\gamma\|_1$, the set $J_\gamma^\varepsilon := \{k \in \mathbb{N} \mid 1 \leq k \leq N_\gamma\}$ contains I_γ^ε and therefore satisfies (27). If in addition $\#J_\gamma^\varepsilon = N_\gamma \leq N < +\infty$, then

$$\left(\sum_{k \in J_\gamma^\varepsilon} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^2 \right)^{1/2} \approx \left(\sum_{k \in J_\gamma^\varepsilon} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^p \right)^{1/p},$$

with a constant that depends on N . □

Finally we derive a variant of Theorem 5 where, under an inner regularity assumption on the family of symbols, we renormalize the frame of eigenfunctions so that each frame element has norm 1. To this end we first prove the following lemma which may be of independent interest.

Lemma 1. *Let $h \in L^2(\mathbb{R}^d)$ and let $\Omega \subseteq \mathbb{R}^d$ have non-empty interior. Then the time-frequency localization operator $H_\Omega^h : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$,*

$$H_\Omega^h f(t) = \int_\Omega \mathcal{V}_h f(x, \xi) h(t - x) e^{2\pi i \xi t} dx dw, \quad (t \in \mathbb{R}^d).$$

has infinite rank.

Proof. The proof is based on the fact that the short-time Fourier transform of Hermite functions are weighted polyanalytic functions (cf. [2, 1, 3]) and therefore cannot vanish on a ball.

Suppose, for the sake of contradiction, that H_Ω^h has rank $n - 1 < +\infty$. Let $h_1, \dots, h_n \in L^2(\mathbb{R}^d)$ be multi-dimensional Hermite functions of order $\leq n$. For example, if $g_1, \dots, g_n \subseteq L^2(\mathbb{R})$ are the first one-dimensional Hermite functions, we can let $h_k \in L^2(\mathbb{R}^d)$ be the tensor product $h_k(x_1, \dots, x_d) := g_k(x_1)g_1(x_2) \dots g_1(x_d)$. Let S_n be the subspace of $L^2(\mathbb{R}^d)$ spanned by h_1, \dots, h_n .

Since S_n has dimension n , it follows that there exists some nonzero $f = \sum_{k=1}^n c_k h_k \in S_n$ such that $H_\Omega^h f = 0$. Consequently,

$$0 = \langle H_\Omega^h f, f \rangle = \int_\Omega |V_h f(z)|^2 dz,$$

and therefore $V_h f \equiv 0$ on Ω . With the notation $(x, w) =: z \in \mathbb{C}^n$ and $m(z) = e^{-x \cdot w i + \pi |z|^2}$, let $F(z) := m(z) V_h f(z) = \overline{V_h f(-\bar{z})}$, it follows that F vanishes on $\Omega' := -\bar{\Omega}$. We will show that $F \equiv 0$. Since m never vanishes, this will imply that $f \equiv 0$, thus yielding a contradiction.

The function $F(z) = \sum_k c_k m(z) V_{h_k} h(-z)$ is a polyanalytic function of order (at most) n (i.e. $(\partial/\partial \bar{z})^n F \equiv 0$ (see [2, 1, 3])). A polyanalytic function that vanishes on a set of non-empty interior must vanish identically. For $d = 1$ this can be proved directly by induction on n or deduced from much sharper uniqueness results (see [5]). The case of general dimension d reduces to $d = 1$ by fixing $d - 1$ variables of F and applying the one-dimensional result. □

We now obtain a variant of Theorem 5.

Theorem 6. *Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of non-negative symbols on \mathbb{R}^{2d} such that $\sum_\gamma \eta_\gamma \approx 1$ and v a w -moderated weight. Assume in addition that there exists a ball B_r and a constant $c > 0$ such that*

$$(28) \quad \eta_\gamma(z) \geq c\chi_{B_r}(z - \gamma) \geq 0 \quad (z \in \mathbb{R}^{2d}, \gamma \in \Gamma).$$

Then, there exists a constant $\alpha > 0$ such that, for every choice of finite subsets of eigenfunctions $\{\phi_k^\gamma \mid \gamma \in \Gamma, 1 \leq k \leq N_\gamma\}$ with $\alpha\|\eta_\gamma\|_1 \leq N_\gamma \leq N < +\infty$, the following frame estimates hold simultaneously for all $1 \leq p \leq +\infty$, with the usual modification for $p = \infty$:

$$(29) \quad \|f\|_{M_v^p} \approx \left(\sum_\gamma \sum_{k=1}^{N_\gamma} |\langle f, \phi_k^\gamma \rangle|^p v(\gamma)^p \right)^{1/p}.$$

Moreover, α can be chosen uniformly for any class of weights v having a uniform constant C_v (cf. (20)).

Remark 13. *If $\{\Omega_\gamma : \gamma \in \Gamma\}$ satisfies (3) and covers \mathbb{R}^{2d} with a bounded number of overlaps, then the corresponding family of characteristic functions η_γ meets the conditions of Theorem 6 (see also Remark 12). This shows that Theorem 1, in Section 1, is a particular case of Theorem 6.*

Remark 14. *The frame in Theorem 6 comprises the first N_γ elements of each of the orthonormal sets $\{\phi_k^\gamma : k \geq 1\}$. These first n_γ functions are the ones that are best concentrated, according to the weight n_γ . This resembles the problem studied in [42]. However the results there require very precise information on the frames being pieced together and hence do not apply here.*

Remark 15. *In the language of [40, 10], this shows that the subspaces spanned by the finite families of eigenfunctions form a stable splitting or fusion frame. From an application point of view, it is useful to dispose of orthogonal projections onto subspaces with time-frequency concentration in a prescribed area of the time-frequency plane.*

Proof of Theorem 6. By Theorem 5, we have that,

$$\|f\|_{M_v^p} \approx \left(\sum_\gamma \sum_{k=1}^{N_\gamma} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^p v(\gamma)^p \right)^{1/p}.$$

Hence, it suffices to show that $\lambda_k^\gamma \approx 1$, for $1 \leq k \leq N_\gamma$.

The upper bound follows from the well-spreadness condition. If (Γ, Θ, g) is an envelope for $\{H_{\eta_\gamma} : \gamma \in \Gamma\}$, then all the singular values of H_{η_γ} are bounded by $\|H_{\eta_\gamma}\|_{2 \rightarrow 2} \leq \|g\|_\infty$, cf. (16).

By Lemma 1, the localization operator $H_{\chi_{B_r}}$ has infinite rank. Hence, the non-zero eigenvalues of $H_{\chi_{B_r}}$ form an infinite non-increasing sequence $\lambda_k^R > 0$, $k \geq 1$. From (28) it follows that $H_{\eta_\gamma} \geq cH_{\chi_{B_r}}$ (cf. Prop. 3) and consequently, for $1 \leq k \leq N_\gamma$, $\lambda_k^\gamma \geq c\lambda_k^R \geq c\lambda_{N_\gamma}^R \geq c\lambda_N^R > 0$. \square

Remark 16. *For the results in this section, the abstract setting of Section 2 allows for the replacement of ℓ_v^p by more general normed spaces. Indeed, the results derived in the abstract setting cover modulation spaces defined with respect to general translation-invariant solid spaces, cf. [23].*

5. FRAMES OF EIGENFUNCTIONS: GENERAL L^2 ESTIMATES

The arguments presented in Section 4.1 make use of the general almost-orthogonality principle in Theorem 2 and carry over to the Banach-space setting by exploiting spectral-invariance results for pseudo-differential operators. In the abstract setting of Section 3 we can still carry out the proofs of Section 4.1 to obtain L^2 estimates. We briefly present these results here. The proofs are, mutatis mutandis, the same as in Section 4 and will be just sketched.

Let $E = L^2(\mathcal{G})$ and let us assume that (A1), (A2) and (A3) from Section 2.2 hold. Let a well-spread family $\{\eta_\gamma\}_\gamma$ of non-negative functions on \mathcal{G} be given. We consider the corresponding family of phase-space multipliers,

$$M_{\eta_\gamma} f = P(\eta_\gamma f), \quad (f \in S_{L^2}).$$

Since each η_γ is non-negative and belongs to $L^1(\mathcal{G})$, according to Proposition 3, the corresponding operator $M_{\eta_\gamma} : S_{L^2} \rightarrow S_{L^2}$ is positive and trace-class, and $\text{trace}(M_{\eta_\gamma}) \lesssim \|\eta_\gamma\|_1$.

Let $M_{\eta_\gamma} : S_{L^2} \rightarrow S_{L^2}$ be diagonalized as $M_{\eta_\gamma} f = \sum_k \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma$, where $\{\phi_k^\gamma \mid k \geq 1\}$ is an orthonormal subset of S_{L^2} and define $M_{\eta_\gamma}^\varepsilon f = \sum_{k: \lambda_k^\gamma > \varepsilon} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma$. When M_{η_γ} has finite rank, $\lambda_k^\gamma = 0$, for $k \gg 1$ and the choice of the corresponding eigenfunction is arbitrary.

We have the following version of the Theorems 4 and 5.

Theorem 7. *Let $\{\eta_\gamma\}_\gamma$ be a well-spread family of non-negative symbols such that $\sum_\gamma \eta_\gamma \approx 1$. Then, there exists constants $0 < c \leq C < +\infty$ such that for all sufficiently small $\varepsilon > 0$,*

$$c\|f\|_2^2 \leq \sum_\gamma \|M_{\eta_\gamma}^\varepsilon f\|_2^2 \leq C\|f\|_2^2, \quad (f \in S_{L^2}).$$

Furthermore, there exists a constant $\alpha > 0$ such that, for every choice of $\alpha\|\eta_\gamma\|_1 \leq N_\gamma \leq N < +\infty$, the family

$$(30) \quad \{\lambda_k^\gamma \phi_k^\gamma \mid \gamma \in \Gamma, 1 \leq k \leq N_\gamma\},$$

is a frame of S_{L^2} .

Remark 17. Note again that when η_γ is the characteristic function of a set Ω_γ , we are picking $\approx |\Omega_\gamma|$ eigenfunctions from each localization operator (phase-space multiplier). Here, $|\Omega_\gamma|$ is the Haar measure of Ω_γ .

Remark 18. The operator M_{η_γ} may have finite rank (for example if \mathcal{G} is a discrete group and η_γ is the characteristic function of a finite set). In this case the choice of the eigenfunctions associated to the singular value zero is irrelevant, since in (30) these are multiplied by zero.

Proof of Theorem 7. This parallels the proofs in Section 4.1 and wraps up their main steps. Since $\sum_\gamma M_{\eta_\gamma} = M_{\sum_\gamma \eta_\gamma}$ and $\sum_\gamma \eta_\gamma \geq A > 0$, it follows from Prop. 3 that $\sum_\gamma M_{\eta_\gamma}$ is positive definite and therefore invertible. Theorem 2 consequently yields,

$$(31) \quad \sum_\gamma \|M_{\eta_\gamma} f\|_{L^2(\mathcal{G})}^2 \approx \|f\|_2^2, \quad (f \in S_{L^2}).$$

In addition, Proposition 5, Corollary 1 and a second application of Theorem 2 yield,

$$(32) \quad \sum_\gamma \|M_{\eta_\gamma}^2 f\|_{L^2(\mathcal{G})}^2 \approx \|f\|_2^2, \quad (f \in S_{L^2}).$$

The thresholded operators $M_{\eta_\gamma}^\varepsilon$ satisfy,

$$(33) \quad \|M_{\eta_\gamma}^\varepsilon f\|_2 \leq \|M_{\eta_\gamma} f\|_2 \leq \|M_{\eta_\gamma}^\varepsilon f\|_2 + \varepsilon \|f\|_2, \quad (f \in S_{L^2}).$$

Applying this to $M_{\eta_\gamma} f$ and noting that M_{η_γ} and $M_{\eta_\gamma}^\varepsilon$ commute gives,

$$(34) \quad \|M_{\eta_\gamma}^2 f\|_2 \leq \|M_{\eta_\gamma}^\varepsilon M_{\eta_\gamma} f\|_2 + \varepsilon \|M_{\eta_\gamma} f\|_2,$$

$$(35) \quad = \|M_{\eta_\gamma} M_{\eta_\gamma}^\varepsilon f\|_2 + \varepsilon \|M_{\eta_\gamma} f\|_2,$$

$$(36) \quad \lesssim \|M_{\eta_\gamma}^\varepsilon f\|_2 + \varepsilon \|M_{\eta_\gamma} f\|_2, \quad (f \in S_{L^2}).$$

Putting all these inequalities together gives,

$$\left(\sum_\gamma \|M_{\eta_\gamma}^\varepsilon f\|_2\right)^{1/2} \lesssim \|f\|_2 \lesssim \left(\sum_\gamma \|M_{\eta_\gamma}^\varepsilon f\|_2\right)^{1/2} + \varepsilon \|f\|_2, \quad (f \in S_{L^2}).$$

This implies that for $0 < \varepsilon \ll 1$,

$$\left(\sum_\gamma \|M_{\eta_\gamma}^\varepsilon f\|_2\right)^{1/2} \approx \|f\|_2,$$

as claimed. The fact that the system in (30) is a frame of S_{L^2} now follows like in Theorem 5, this time using Proposition 3 to estimate,

$$\#\{\lambda_k^\gamma : \lambda_k^\gamma > \varepsilon\} \leq \varepsilon^{-1} \text{trace}(M_{\eta_\gamma}) \lesssim \varepsilon^{-1} \|\eta_\gamma\|_1,$$

□

5.1. Application to time-scale analysis. We now show how to apply Theorem 7 to time-scale analysis. Let $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ be a Schwartz-class radial function with several vanishing moments. The *wavelet transform* of a function $f \in L^2(\mathbb{R}^d)$ with respect to ψ is defined by,

$$(37) \quad W_\psi f(x, s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) \overline{\psi\left(\frac{t-x}{s}\right)} dt, \quad (x \in \mathbb{R}^d, s > 0).$$

If ψ is properly normalized (and we assume so thereof), W_ψ maps $L^2(\mathbb{R}^d)$ isometrically into $L^2(\mathbb{R}^d \times \mathbb{R}_+, s^{-(d+1)} dx ds)$. For a (measurable) bounded symbol $m : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{C}$, the wavelet multiplier $\text{WM}_m : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is defined by

$$(38) \quad \text{WM}_m f(t) = \int_0^{+\infty} \int_{\mathbb{R}^d} m(x, s) W_\psi f(x, s) \pi(x, s) \psi(t) dx \frac{ds}{s^{d+1}},$$

where $\pi(x, s)\psi(t) := s^{-d/2}\psi\left(\frac{t-x}{s}\right)$. (Here, the integral converges in the weak sense.) Note that $\text{WM}_m = W_h^*(m W_h F)$.

In order to apply the model from Section 2.2 we consider the affine group $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}_+$, where multiplication is given by $(x, s) \cdot (x', s') = (x + sx', ss')$. We let $E := L^2(\mathcal{G})$ and $S_E := W_\psi L^2(\mathbb{R}^d)$. In complete analogy to the time-frequency analysis case, we let $P := W_\psi (W_\psi)^* : L^2(\mathcal{G}) \rightarrow W_\psi L^2(\mathbb{R}^d)$ be the orthogonal projection and $K := |W_\psi \psi|$. We further let $w(x, s) := \max\{1, s^d\}$. The kernel K belongs to $W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$ if ψ has sufficiently many vanishing moments (see [34, Section 4.2]).

As an example of a well-spread family of symbols we consider the characteristic functions of a cover of \mathcal{G} by irregular boxes. Let us take as centers the points

$$\Gamma := \{\gamma_{j,k} := (k2^j, 2^j) \mid j \in \mathbb{Z}, k \in \mathbb{Z}^d\},$$

and consider a family of boxes around $(0, 1) \in \mathbb{R}^d \times \mathbb{R}_+$,

$$V_{j,k} := [-a_{j,k}^1/2, a_{j,k}^1/2] \times \dots \times [-a_{j,k}^d/2, a_{j,k}^d/2] \times [(b_{j,k})^{-1}, b_{j,k}],$$

where $0 \leq a_{j,k}^l \leq a < +\infty$, $l = 1, \dots, d$ and $0 < b^{-1} \leq b_{j,k} \leq b < +\infty$. Let us set $U_{k,j} := \gamma_{j,k} V_{j,k}$. Then the family of characteristic functions $\{\chi_{U_{k,j}}\}_{j,k}$ is well-spread. The corresponding operators $\text{WM}_{k,j} := \text{WM}_{\chi_{U_{k,j}}}$ are known as *wavelet localization operators* [14, 16, 15]. Let us denote again by $\{\phi_k^\gamma \mid k \geq 1\}$ an eigenset of $\text{WM}_{k,j}$ ordered according to the corresponding eigenvalues λ_k^γ .

Noting that $\|\chi_{k,j}\|_1 = |U_{k,j}| = |V_{k,j}| = \frac{1}{d} \prod_{j=1}^d a_{j,k}^j \cdot [(b_{j,k})^d - (b_{j,k})^{-d}]$, Theorem 7 yields,

Theorem 8. *Assume that $\{U_{j,k} \mid j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ covers $\mathbb{R}^d \times \mathbb{R}_+$. Then there exists a constant $\alpha > 0$ such for every choice of $\alpha |U_{k,j}| \leq N_\gamma \leq N < +\infty$, the family*

$$\{\lambda_k^\gamma \phi_k^\gamma \mid \gamma \in \Gamma, 1 \leq k \leq N_\gamma\},$$

is a frame of $L^2(\mathbb{R}^d)$.

5.2. Application to Gabor analysis. We now consider as \mathcal{G} discrete subgroups of \mathbb{R}^{2d} , more precisely, *lattices* of the form $\Lambda = P\mathbb{Z}^{2d}$, where $P \in \mathbb{R}^{2d \times 2d}$ is an invertible matrix. This choice leads to Gabor analysis and in particular to the setting of *Gabor frames*. Here, the corresponding phase-space multipliers are Gabor multipliers, [25, 19, 33]. We now show how to apply the model from Section 2. This is completely analogous to the treatment of continuous time-frequency expansions in Section 4.

Given a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ and a window $\varphi \in M^1(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$ we consider the Gabor system $F(\varphi, \Lambda) = \{\pi(\lambda)\varphi, \lambda \in \Lambda\}$ of time-frequency shifted windows, i.e., for $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, $\pi(\lambda)\varphi(t) =$

$\varphi(t - \lambda_1)e^{2\pi i\lambda_2 t}$. We assume that $F(\varphi, \Lambda)$ is a tight frame for $L^2(\mathbb{R}^d)$. This means that for some constant $A > 0$, every function $f \in L^2(\mathbb{R}^d)$ has an expansion,

$$f = A \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\varphi \rangle \pi(\lambda)\varphi.$$

The corresponding analysis operator $\mathcal{V}_{\Lambda, \varphi} : L^2(\mathbb{R}^d) \rightarrow \ell^2(\Lambda)$ is given by $\mathcal{V}_{\Lambda, \varphi} f(\lambda) = \sqrt{A} \langle f, \pi(\lambda)\varphi \rangle$. Let $S_{\ell^2} = \mathcal{V}_{\Lambda, \varphi}(L^2(\mathbb{R}^d))$. The orthogonal projection $P : \ell^2(\Lambda) \rightarrow S_{\ell^2}$ is $P = \mathcal{V}_{\Lambda, \varphi} \mathcal{V}_{\Lambda, \varphi}^*$ and is therefore represented by the matrix $\kappa(\mu, \lambda) = A \langle \pi(\mu)\varphi, \pi(\lambda)\varphi \rangle$. Consequently,

$$|\kappa(\mu, \lambda)| = A |\langle \pi(\mu)\varphi, \pi(\lambda)\varphi \rangle| = A |\mathcal{V}_{\Lambda, \varphi} \varphi(\mu - \lambda)|, \quad (\mu, \lambda \in \Lambda).$$

Since $\varphi \in M^1(\mathbb{R}^d)$, $\mathcal{V}_{\Lambda, \varphi}$ maps $M^1(\mathbb{R}^d)$ into $\ell^1(\Lambda)$ (see for example [26]) and we conclude that $K := |\mathcal{V}_{\Lambda, \varphi} \varphi| \in \ell^1(\Lambda)$. Hence, (A1), (A2) and (A3) from Section 2 are satisfied with $\mathcal{G} = \Lambda$, $E = \ell^2$ and $w \equiv 1$.

For a bounded sequence $m : \Lambda \rightarrow \mathbb{C}$, the Gabor multiplier $GM_m : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is defined by applying m to the frame expansion

$$GM_m f = A \sum_{\lambda} m(\lambda) \langle f, \pi(\lambda)\varphi \rangle \pi(\lambda)\varphi.$$

Hence, $GM_m f = \mathcal{V}_{\Lambda, \varphi}^*(Am\mathcal{V}_{\Lambda, \varphi} f)$ and $V_{\varphi} GM_m V_{\varphi}^* : S_{\ell^2} \rightarrow S_{\ell^2}$ is a phase-space multiplier with symbol Am .

We may now consider a well-spread family $\{\eta_{\gamma}\}_{\gamma}$ of non-negative symbols defined on \mathbb{R}^{2d} with $\sum_{\gamma} \eta_{\gamma} \approx 1$ and restrict each η_{γ} to Λ . Then, for $\gamma \in \Gamma$, we let $\{\phi_k^{\gamma} : k \geq 1\}$ be the set of eigenfunctions of $GM_{\eta_{\gamma}}$ in decreasing order with respect to the corresponding eigenvalues λ_k^{γ} . Then, Theorem 7 yields the following result.

Theorem 9. *There exists a constant $\alpha > 0$ such for every choice of $N_{\gamma} \geq 1$ for which*

$$\alpha \|\eta_{\gamma}\|_1 \leq N_{\gamma} \leq N < +\infty,$$

the family $\{\lambda_k^{\gamma} \phi_k^{\gamma} \mid \gamma \in \Gamma, 1 \leq k \leq N_{\gamma}\}$ is a frame of $L^2(\mathbb{R}^d)$.

APPENDIX: PROOF OF THEOREM 2

In this appendix we prove Theorem 2. The proof is essentially contained in [43], but is not explicitly stated in the required generality. We therefore show how to derive Theorem 2 from some technical lemmas in [43].

Suppose that Assumptions (A1) and (A2) from Section 2.2 hold. Let $\{T_{\gamma} \mid \gamma \in \Gamma\}$ be a well-spread family of operators with envelope (Γ, Θ, g) . According to the definition, $g \in W_R(L^{\infty}, L_w^1)(\mathcal{G})$. The article [43] considers a technical variant of this space, the weak amalgam space $W_R^{\text{weak}}(L^{\infty}, L_w^1)$ (see [43, Section 2.4]), which we do not wish to introduce here. By [43, Lemma Prop. 1], $L^1(\mathcal{G}) \hookrightarrow W_R^{\text{weak}}(L^{\infty}, L_w^1)(\mathcal{G}) \hookrightarrow W_R(L^{\infty}, L_w^1)(\mathcal{G})$. We will use certain results from [43] that are proved under the thus more general hypothesis $g \in W_R^{\text{weak}}(L^{\infty}, L_w^1)(\mathcal{G})$. Nonetheless we point out that, according to [43, Lemma Prop. 1], for certain groups $W_R^{\text{weak}}(L^{\infty}, L_w^1)(\mathcal{G})$ is just $L^1(\mathcal{G})$. This is the case for example for $\mathcal{G} = \mathbb{R}^d$. Hence, in Section 4, the hypothesis $g \in W(L^{\infty}, L_w^1)$ could be relaxed to $g \in L^1$.

We consider an L^2 -valued version of $E_d(\Gamma)$,

$$E_{d, L^2} = E_{d, L^2}(\Gamma) := \{f : \Gamma \rightarrow L^2(\mathcal{G}) \mid (\|f_{\gamma}\|_{L^2})_{\gamma \in \Gamma} \in E_d(\Gamma)\},$$

and endow it with the norm $\|(f_{\gamma})_{\gamma \in \Gamma}\|_{E_{d, L^2}} := \|(\|f_{\gamma}\|_{L^2})_{\gamma \in \Gamma}\|_{E_d}$. Let $U \subseteq \mathcal{G}$ be a relatively compact neighborhood of the identity. Consider the operators C_T and S_U formally defined by,

$$\begin{aligned} C_T(f) &:= (T_{\gamma}(f))_{\gamma \in \Gamma}, \quad (f \in S_E), \\ S_U((f_{\gamma})_{\gamma \in \Gamma}) &:= \sum_{\gamma} P(f_{\gamma}) \chi_{\gamma U}, \quad (f_{\gamma} \in L^2(\mathcal{G})), \end{aligned}$$

where $\chi_{\gamma U}$ denotes the characteristic function of the set γU . These operators satisfy the following mapping properties.

Proposition 7.

- (a) *The analysis operator C_T maps S_E boundedly into $E_{d, L^2}(\Gamma)$.*

- (b) For every relatively compact neighborhood of the identity U , and every sequence $F \equiv (f_\gamma)_\gamma \in E_{d,L^2}$, the series defining $S_U(F)$ converge absolutely in $L^2(\mathcal{G})$ at every point. Moreover, the operator S_U maps $E_{d,L^2}(\Gamma)$ boundedly into E (with a bound that depends on U).

Proof. Part (b) is proved in [43, Prop. 4 (b)]. Part (a) is a slight variant of [43, Prop. 4 (a)]; for completeness we give a full argument.

Let $f \in S_E$. Since η_γ is bounded, $f\eta_\gamma \in E$. By the definition of well-spread family (cf. (17)),

$$\begin{aligned} |T_\gamma f(x)| &\leq \int_{\mathcal{G}} |f(y)| g(\gamma^{-1}y) \Theta(y^{-1}x) dy \\ &= (|f| L_\gamma g) * \Theta(x). \end{aligned}$$

By Young's inequality $L^1 * L^2 \hookrightarrow L^2$ we have,

$$\|T_\gamma f\|_2 \leq \|\Theta\|_2 \int_{\mathcal{G}} |f(y)| g(\gamma^{-1}y) dy \lesssim \|\Theta\|_{W(L^\infty, L_w^1)} \int_{\mathcal{G}} |f(y)| g(\gamma^{-1}y) dy.$$

Now the solidity of E and [43, Lemma 4] (see also [23, Lemma 3.8]) yield,

$$\|C_T(f)\|_{E_{d,L^2}} \lesssim \|f\|_{W(L^\infty, E)} \|g\|_{W_R(L^\infty, L_w^1)}.$$

Finally, by Prop. 2, $\|f\|_{W(L^\infty, E)} \lesssim \|f\|_E$. \square

Remark 19. Note that in the last proof the use of the L^2 norm is somehow arbitrary; a number of other function norms could have been used instead (cf. [43, Prop. 4]).

Now we prove the key approximation result (cf. [43, Theorem 1]).

Theorem 10. Given $\varepsilon > 0$, there exists U_0 , a relatively compact neighborhood of e such that for all $U \supseteq U_0$,

$$(39) \quad \left\| \sum_{\gamma} T_\gamma f - S_U C_T(f) \right\|_E \leq \varepsilon \|f\|_E, \quad (f \in S_E).$$

Remark 20. The neighborhood U_0 can be chosen uniformly for any class of spaces E having the same weight w and the same constant $C_{E,w}$ (cf. (13)).

Concerning the parameters in Assumptions (A1) and (A2) and (17), the choice of U_0 only depends on $\|K\|_{W(L^\infty, L_w^1)}$, $\|K\|_{W_R(L^\infty, L_w^1)}$, $\|\Theta\|_{W(L^\infty, L_w^1)}$, $\|\Theta\|_{W_R(L^\infty, L_w^1)}$, $\|g\|_{W_R(L^\infty, L_w^1)}$ and $\rho(\Gamma)$ (cf. (9)).

Proof. Let $f \in S_E$ and let U be a relatively compact neighborhood of e . Because of the inclusion $S_E \hookrightarrow W(L^\infty, E)$ in Prop. 2, it suffices to dominate the left-hand side of (39) by $\varepsilon \|f\|_{W(L^\infty, E)}$.

Note that since $T_\gamma f \in S_E$, $S_U C_T f(x) = \sum_{\gamma} T_\gamma f(x) \chi_{\gamma U}(x)$. Hence, using (17) let us estimate,

$$\begin{aligned} \left| \sum_{\gamma} T_\gamma f(x) - S_U C_T f(x) \right| &= \left| \sum_{\gamma} \chi_{\gamma(\mathcal{G} \setminus U)}(x) T_\gamma(f)(x) \right| \\ &\leq \sum_{\gamma} \int_{\mathcal{G}} |f(y)| g(\gamma^{-1}y) \Theta(y^{-1}x) \chi_{\gamma(\mathcal{G} \setminus U)}(x) dy. \end{aligned}$$

The rest of the proof is carried out exactly as in [43, Theorem 1]. Indeed, the proof there only depends on the estimate just derived.⁴ (The definition of well-spread family of operators was tailored so that the proof in [43, Theorem 1] would still work.) \square

Finally we can prove Theorem 2.

Proof of Theorem 2. Let $\{T_\gamma : \gamma \in \Gamma\}$ be a well-spread family of operators and suppose that the operator $\sum_{\gamma} T_\gamma : S_E \rightarrow S_E$ is invertible. We have to show that for $f \in S_E$, $\|f\|_E \approx \|C_T(f)\|_{E_{d,L^2}(\Gamma)}$. The estimate $\|C_T(f)\|_{E_{d,L^2}(\Gamma)} \lesssim \|f\|_E$ is proved in Proposition 7 (a). To establish the second inequality, consider the operator $PS_U C_T : S_E \rightarrow S_E$. Then for $f \in S_E$,

$$\left\| \sum_{\gamma} T_\gamma f - PS_U C_T f \right\|_E = \|P \sum_{\gamma} T_\gamma f - PS_U C_T f\|_E \lesssim \left\| \sum_{\gamma} T_\gamma f - S_U C_T f \right\|_E.$$

⁴The function Θ is called H in the proof [43, Theorem 1].

This estimate, together with Theorem 10 implies that $\|\sum_{\gamma} T_{\gamma} - PS_U C_T\|_{S_E \rightarrow S_E} \rightarrow 0$ as U grows to \mathcal{G} . Hence, there exists U such that $PS_U C_T$ is invertible on S_E . Consequently, for $f \in S_E$, $\|f\|_E \approx \|PS_U C_T f\|_E \lesssim \|C_T(f)\|_{E_{d,L^2}(\Gamma)}$. Here we have used the boundedness of S_U - contained in Proposition 7 (b) - and the boundedness of $P : E \rightarrow W(L^\infty, E) \hookrightarrow E$ - contained in Proposition 2. \square

REFERENCES

- [1] L. D. Abreu. On the structure of Gabor and super Gabor spaces. *Monatsh. Math.*, 161(3):237–253, 2010.
- [2] L. D. Abreu. Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions. *Appl. Comput. Harmon. Anal.*, 29(3):287–302, 2010.
- [3] L. D. Abreu and K. Gröchenig. Banach Gabor frames with Hermite functions: polyanalytic spaces from the Heisenberg group. *Applicable Anal.*, to appear.
- [4] A. Aldroubi, C. Cabrelli, and U. Molter. Wavelets on irregular grids with arbitrary dilation matrices, and frame atoms for $L^2(\mathbb{R}^d)$. *Appl. Comput. Harmon. Anal.*, Special Issue on Frames II.:119–140, 2004.
- [5] M. Balk. Polyanalytic functions and their generalizations. Complex analysis I. *Encycl. Math. Sci.* 85, 195–253 (1997); translation from *Itogi Nauki Tekh.*, Ser. Sovrem. Probl. Math., Fundam Napravleniya 85, 187–246 (1991)., 1991.
- [6] P. Boggiatto. Localization operators with L^p symbols on modulation spaces. In *Advances in Pseudo-differential Operators*, volume 155 of *Oper. Theory Adv. Appl.*, pages 149–163. Birkhäuser, Basel, 2004.
- [7] P. Boggiatto and E. Cordero. Anti-Wick quantization with symbols in L^p spaces. *Proc. Amer. Math. Soc.*, 130(9):2679–2685 (electronic), 2002.
- [8] M. Bownik and K.-P. Ho. Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces. *Trans. Amer. Math. Soc.*, 358(4):1469–1510, 2006.
- [9] C. Cabrelli, U. Molter, and J. L. Romero. Non-uniform painless decompositions for anisotropic Besov and Triebel-Lizorkin spaces. Preprint, (arXiv:1108.2748v1 [math.FA]).
- [10] P. G. Casazza and G. Kutyniok. Frames of subspaces. In *Wavelets, Frames and Operator Theory*, volume 345 of *Contemp. Math.*, pages 87–113. Amer. Math. Soc., Providence, RI, 2004.
- [11] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.
- [12] E. Cordero and K. Gröchenig. Symbolic calculus and Fredholm property for localization operators. *J. Fourier Anal. Appl.*, 12(4):371–392, 2006.
- [13] W. Dahmen, S. Dekel, and P. Petrushev. Two-level-split decomposition of anisotropic Besov spaces. *Constr. Approx.*, 31(2):149–194, 2010.
- [14] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, July 1988.
- [15] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 36(5):961–1005, 1990.
- [16] I. Daubechies and T. Paul. Time-frequency localisation operators - a geometric phase space approach: II. The use of dilations. *Inverse Probl.*, 4(3):661–680, 1988.
- [17] M. Dörfler, H. G. Feichtinger, and K. Gröchenig. Time-frequency partitions for the Gelfand triple (S_0, L^2, S_0') . *Math. Scand.*, 98(1):81–96, 2006.
- [18] M. Dörfler and K. Gröchenig. Time-frequency partitions and characterizations of modulations spaces with localization operators. *J. Funct. Anal.*, 260(7):1903 – 1924, 2011.
- [19] M. Dörfler and B. Torrésani. Representation of operators in the time-frequency domain and generalized Gabor multipliers. *J. Fourier Anal. Appl.*, 16(2):261–293, 2010.
- [20] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz. Nagy and J. Szabados. edition, 1983.
- [21] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Canad. J. Math.*, 42(3):395–409, 1990.
- [22] H. G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.
- [23] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2):307–340, 1989.
- [24] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, II. *Monatsh. Math.*, 108(2-3):129–148, 1989.
- [25] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.
- [26] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser Boston.
- [27] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.
- [28] J. J. F. Fournier and J. Stewart. Amalgams of L^p and ℓ^q . *Bull. Amer. Math. Soc., New Ser.*, 13:1–21, 1985.
- [29] K. Gröchenig. Irregular sampling of wavelet and short-time Fourier transforms. *Constr. Approx.*, 9:283–297, 1993.

- [30] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.
- [31] K. Gröchenig. Composition and spectral invariance of pseudodifferential operators on modulation spaces. *J. Anal. Math.*, 98:65–82, 2006.
- [32] K. Gröchenig. Time-Frequency analysis of Sjöstrand’s class. *Rev. Mat. Iberoam.*, 22(2):703–724, 2006.
- [33] K. Gröchenig. Representation and approximation of pseudodifferential operators by sums of Gabor multipliers. *Appl. Anal.*, 90(3-4):385–401, 2011.
- [34] K. Gröchenig and M. Piotrowski. Molecules in coorbit spaces and boundedness of operators. *Studia Math.*, 192(1):61–77, 2009.
- [35] F. Holland. Harmonic analysis on amalgams of L^p and ℓ^q . *J. London Math. Soc.*, 10:295–305, 1975.
- [36] F. Jaillet and B. Torrésani. Time–frequency jigsaw puzzle: adaptive multiwindow and multilayered Gabor expansions. *Int. J. Wavelets Multiresolut. Inf. Process.*, 2:293–316, 2007.
- [37] Y. Liu, A. Mohammed, and M. Wong. Wavelet multipliers on $L^p(\mathbb{R}^n)$. *Proc. Amer. Math. Soc.*, 136(3):1009–1018, 2008.
- [38] M. Liuni, A. Röbel, M. Romito, and X. Rodet. Rényi information measures for spectral change detection. In *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2011*, pages 3824 – 3827, May 2011.
- [39] M. Nashed and Q. Sun. Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. *J. Funct. Anal.*, 258(7):2422–2452, 2010.
- [40] P. Oswald. Stable subspace splittings for Sobolev spaces and domain decomposition algorithms. In *Keyes, David E (ed) et al, Domain Decomposition Methods in Scientific and Engineering Computing Proceedings of the 7th International Conference on Domain Decomposition, October 27-30, 1993, Pennsylvania State University, PA, USA Providence, RI: Ameri.* 1994.
- [41] H. Rauhut. Wiener amalgam spaces with respect to quasi-Banach spaces. *Colloq. Math.*, 109(2):345–362, 2007.
- [42] J. L. Romero. Surgery of spline-type and molecular frames. *J. Fourier Anal. Appl.*, 17:135 – 174, 2011.
- [43] J. L. Romero. Characterization of coorbit spaces with phase-space covers. *J. Funct. Anal.*, 262(1):59–93, 2012.
- [44] C. Schörkhuber and A. Klapuri. Constant-Q toolbox for music processing. In *Proceedings of the 7th Sound and Music Computing Conference (SMC), 2010*, 2010.
- [45] B. Simon. *Trace Ideals and their Applications*. Cambridge University Press, Cambridge, 1979.
- [46] J. Sjöstrand. An algebra of pseudodifferential operators. *Math. Res. Lett.*, 1(2):185–192, 1994.
- [47] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. *Séminaire sur les équations aux Dérivées Partielles, 1994-1995, École Polytech, Palaiseau, Exp. No. IV*, 21, 1995.
- [48] G. A. Velasco, N. Holighaus, M. Dörfler, and T. Grill. Constructing an invertible constant-Q transform with non-stationary Gabor frames. *Proceedings of DAFX11*, Paris, 2011.
- [49] M. Wong. *Localization Operators*. Seoul National University, Seoul, 1999.
- [50] M.-W. Wong. *Wavelet Transforms and Localization Operators*. Operator Theory: Advances and Applications. 136. Basel: Birkhäuser, 2002.

E-mail address: monika.doerfler@univie.ac.at

E-mail address: jose.luis.romero@univie.ac.at

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15,A-1090 WIEN, AUSTRIA